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Positive-Real Structure and High-Gain Adaptive Stabilization

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This paper investigates the use of positive real conditions in the analysis of high-gain adaptive control rules for linear systems subjected to nonlinear perturbations of the state and input.

1. Introduction

THE problem of adaptive stabilization of an m-input m-output linear time-invariant system $S(A, B, C)$ in $\mathbb{R}^{n \times l}$, of the form

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t), \end{aligned}\right\} \tag{1.1}$$

has recently been considered in the case of $l = 1$ (Brynes & Willems, 1984; Mårtenssen, 1986) and $l \geqslant 1$ (Ilchmann, Owens, & Prätzel-Wolters, 1986). More precisely, the problem of feedback stabilization of (1.1) in the situation of unknown A, B, C and state dimension n has been considered using the time-varying feedback

$$u(t) = -k(t)y(t) \tag{1.2}$$

where $k(t)$ is a time-varying gain, generated from measurement data. We impose the structural constraints that

(i) $S(A, B, C)$ is minimum-phase, and
(ii) the spectrum $\sigma(CB)$ lies in the open right half complex plane $\mathbb{C}^+$.

The original result of Byrnes & Willems proved that, with $l = 1$, the adaptive gain law

$$\dot{k}(t) = y^{\mathrm{T}}(t)y(t), \qquad k(0) = k_0 \tag{1.3}$$

will generate a stable closed-loop response $y(\bullet) \in \mathrm{L}_2^m[0, \infty)$ for any initial state $x_0 \in \mathbb{R}^n$ and initial gain $k_0 \in \mathbb{R}$, while ensuring that the gain variation $k(\bullet)$ is bounded and convergent in the sense that

$$\lim_{t\to\infty} k(t) = k_\infty < \infty. \tag{1.4}$$

The work of Mårtenssen (1986) provided some generalizations of this result, while the recent work of Ilchmann, Owens, & Prätzel-Wolters (1986) generated a large class of gain adaptation laws that (a) enable the designer to guarantee solutions in any choice of finite intersections of $L_p[0, \infty)$ spaces ($p \geq 1$) and (b) are tolerant to 'small' finite-gain memoryless nonlinearities in the state $x(\bullet)$.

In this paper, we provide an alternative proof of previous results, by relating the possibility of 'gain divergence' to the system—theoretic criterion for positive-real matrices (Anderson, 1967). This proof not only provides a 'physical' mechanism for explaining the convergence of the adaptive scheme, but also extends previous norm-based approaches by permitting, under well-defined circumstances, the inclusion of stability in the presence of unbounded nonlinear elements.

In Section 2, the problem to be considered is defined. In Section 3, the fundamental theorems describing the behaviour of trajectories in situations of gain divergence are derived. In Section 4, the results are combined with those of Ilchmann *et al.* (1986), to demonstrate convergence of a wide class of adaptive gain mechanisms. In Section 5, the notion of (gain-dependent) switching functions introduced by Willems & Byrnes (1984) and Nussbaum (1983) is generalized to allow switching as a function of both current and past gain and input data. The generalization permits a wide range of previously unknown gain adaptation mechanisms to be introduced. Finally, in Section 6, the switching-mechanisms proof is discussed as a device for further extensions to the material of Section 4.

2. Problem definition and notation

Throughout the paper, we will denote $\mathbb{C}^\alpha$ to be the region of the complex plane defined by

$$\mathbb{C}^\alpha = \{s : \operatorname{Re} s < \alpha\}, \qquad \mathbb{C}^+ = \{s : \operatorname{Re} s > 0\}, \qquad \mathbb{C}^- = \{s : \operatorname{Re} s < 0\}.$$

The basic problem to be considered is the stabilization of the linear system $S(A, B, C)$ defined by (1.1) subject to the structural constraints:

(i) $S(A, B, C)$ is controllable, observable, and minimum-phase with all zeros lying in the region $\mathbb{C}^{-\lambda_0}$ for some $\lambda_0 > 0$;

(ii) CB has spectrum

$$\sigma(CB) \subset \mathbb{C}^+. \tag{2.1}$$

If $S(A, B, C)$ possesses these properties, it will be termed of class $\Sigma(\lambda_0)$. Clearly, if $S(A, B, C)$ is of class $\Sigma(0)$, it is of class $\Sigma(\lambda_0)$ for some $\lambda_0 > 0$, and vice versa.

The stabilization is required to be adaptive in the sense that $k(t)$ is generated from a causal map $\psi : z(\bullet) \mapsto k(\bullet)$ of data records of a measurement vector

$$z(t) = F(x(T)) \in \mathbb{R}^{q \times l} \tag{2.2}$$

where F is a linear or nonlinear piecewise continuous function of finite incremental gain F_0 i.e. $\|F(\boldsymbol{x})\| \leqslant F_0 \|\boldsymbol{x}\| \ \forall \boldsymbol{x} \in \mathbb{R}^{n \times l}$. Here ψ is assumed to be independent of A, B, C, λ_0, and state dimension n, and hence stabilizes all systems of class $\Sigma(0)$ despite ignorance of the details of plant dynamics. An example of a solution to this problem is the map ψ defined by (1.3) with $l = 1$ and $z(t) = y(t)$. For design purposes, however, it is of interest to have available a wider class of adaptive mechanisms. The first problem considered in the following sections is the use of positive-real conditions, and their consequences (Anderson, 1967) for the existence of a particular solution of a Liapunov equation, in the characterization of such a class of adaptive mechanisms. Despite the wide choice of mechanism available, it is also important to be able to assess the robustness of the feedback schemes. As will be seen, the use of positive-real analysis is well suited to this task, since it permits the demonstration that the closed-loop system is robust with respect to a wide class of nonlinear time-varying perturbations which can include *unbounded* gain elements.

To define the problem of robustness with respect to nonlinear time-varying perturbations more precisely, we will consider the linear system (1.1) of class $\Sigma(0)$ perturbed to produce the nonlinear time-varying system $S(A, B, C, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{f})$ in $\mathbb{R}^{n \times l}$ described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{g}(\boldsymbol{x}(t), t) + B[\boldsymbol{u}(t) + \boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{y}(t), t) - \boldsymbol{f}(\boldsymbol{y}(t), t)],$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^{n \times l}. \tag{2.3a,b,c}$$

Such a system will be termed *of class* $\mathrm{C}(g_0, h_1, h_2, h_3)$ iff, for all $t \geqslant 0$, $\boldsymbol{x} \in \mathbb{R}^{n \times l}$, and $\boldsymbol{u}, \boldsymbol{y} \in \mathbb{R}^{m \times l}$, the following inequalities hold for $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{f}$:

$$\|\boldsymbol{g}(\boldsymbol{x}, t)\| \leqslant g_0 \|\boldsymbol{x}\|, \tag{2.4a}$$
$$\operatorname{tr}[\boldsymbol{y}^{\mathrm{T}} CB\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}, t)] \leqslant \|\boldsymbol{y}\|_2 (h_1 \|\boldsymbol{x}\| + h_2 \|\boldsymbol{u}\|_2 + h_3 \|\boldsymbol{y}\|_2), \tag{2.4b}$$
$$\operatorname{tr}[\boldsymbol{y}^{\mathrm{T}} CB\boldsymbol{f}(\boldsymbol{y}, t)] \geqslant 0, \tag{2.4c}$$

where $\mathbb{R}^{n \times l}$ is assumed to be endowed with the norm $\|\bullet\|$, and $\|\bullet\|_2$ denotes the usual Euclidean norm $\|\boldsymbol{x}\|_2 = [\operatorname{tr}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})]^{\frac{1}{2}}$. Note that any linear system of class $\Sigma(0)$ is also of class $\mathrm{C}(0, 0, 0, 0)$. To complete this section, we make the following comments on the form of nonlinearity assumed:

(a) The term $\boldsymbol{g}(\boldsymbol{x}, t)$ represents time-varying, linear or nonlinear, state-dependent perturbations to the term $A\boldsymbol{x}$ of uniformly bounded finite gain.

(b) The terms $\boldsymbol{h}$ and $\boldsymbol{f}$ represent state, input and output, time-varying linear or nonlinear perturbations to the plant input $\boldsymbol{u}$ due to, for example, feedback loops within the process dynamics or nonlinear effects in the plant sensors or actuators. Equations (2.4b,c) bound the growth of these nonlinearities and, it should be noted, permit 'unbounded' effects of the correct sign.

3. Theorems on gain divergence: a positive-real approach

In this section, we consider the system (2.3) of class $\mathrm{C}(g_0, h_1, h_2, h_3)$ subjected to the control (1.2) where $k(t)$ is an arbitrary monotonically nondecreasing piecewise continuous function satisfying the divergence condition.

$$\lim_{t \to \infty} k(t) = +\infty. \tag{3.1}$$

A crucial structural property underlying the proof of our results is that of positive-realness. An $m \times m$ transfer-function matrix $H(s)$ is called *positive-real* iff (Anderson, 1967)

(a) its elements are analytic for $s \in \mathbb{C}^+$;
(b) $\overline{H(s)} = H(\bar{s})$ for $s \in \mathbb{C}^+$;
(c) $H(s) + H^{\mathrm{T}}(\bar{s})$ is nonnegative definite for $s \in \mathbb{C}^+$.

The results of Anderson (1967) can now be stated in the form of the following lemma.

LEMMA 1 *Let $H(\bullet)$ be an $m \times m$ transfer function matrix generated from an asymptotically stable, controllable and observable linear time-invariant state-space model $S(A, B, C)$ in $\mathbb{R}^{n \times l}$. Then $H(\bullet)$ is positive-real iff there exists a symmetric positive definite matrix P and a matrix L such that the following Liapunov-like equation is satisfied:*

$$A^{\mathrm{T}}P + PA = -LL^{\mathrm{T}},$$

with $PB = C^{\mathrm{T}}$.

Remark. Anderson's original result is stated for systems in $\mathbb{R}^n$. The 'extension' to systems in $\mathbb{R}^{n \times l}$ is trivial.

Lemma 1 is not quite in the form required for this paper. It is therefore rephrased in the following form.

LEMMA 2 *Let H be as defined in Lemma 1 with all poles and zeros in the region $\mathbb{C}^{-\lambda_0}$ (with $\lambda_0 > 0$) in the complex plane. If the $m \times m$ transfer-function matrix $H_0(s)$ is given by the shift operation*

$$H_0(s) \equiv H(s - \lambda_0), \tag{3.2}$$

then H_0 is asymptotically stable with minimal realization $S(A + \lambda_0 I, B, C)$. Moreover, if $H_0(\bullet)$ is positive-real, then there exists symmetric positive definite matrices P and Q such that

$$A^{\mathrm{T}}P + PA = -Q, \qquad PB = C^{\mathrm{T}}. \tag{3.3}$$

Proof. The proof of stability and the form of the minimal realization is trivial and hence omitted for brevity. We can now use Lemma 1 to guarantee the existence of a symmetric positive definite matrix P and a matrix L such that $PB = C^{\mathrm{T}}$ and

$$(A + \lambda_0 I)^{\mathrm{T}}P + P(A + \lambda_0 I) = -LL^{\mathrm{T}};$$

this is just (3.3), as required, with $Q = LL^{\mathrm{T}} + 2\lambda_0 P = Q^{\mathrm{T}} > 0$. □

The relationship of the positive-real condition to high-gain studies is expressed by the following lemma.

LEMMA 3 *Let the system* (1.1) *be controllable, observable, and minimum-phase,* *with all zeros in the region $\mathbb{C}^{-\lambda_0}$ (for some $\lambda_0 > 0$) in the complex plane, and suppose that CB is positive definite in the sense that*

$$N := CB + (CB)^{\mathrm{T}} > 0. \tag{3.4}$$

Then $S(A, B, C)$ is of class $\Sigma(\lambda_0)$ and, under unity negative feedback control $\boldsymbol{u}(t) = \boldsymbol{r}(t) - k\boldsymbol{y}(t)$, with k a scalar constant gain, there exists $k^* \in \mathbb{R}^+$ such that the closed-loop system $S(A - kBC, B, C)$ is asymptotically stable, for all $k \geqslant k^*$, with all poles and zeros in the region $\mathbb{C}^{-\lambda_0}$ of the complex plane. Further, if the closed-loop transfer-function matrix is factorized into the form $H(s)CB$, and $H_0(s)$ is defined by (3.2), then H_0 is asymptotically stable, and we can choose k^* to ensure that H_0 is positive-real for all $k \geqslant k^*$.

Proof. Condition (3.4) implies condition (2.1) which, together with the minimum-phase assumption, implies the first part of the result using basic properties of multivariable root loci (Owens, 1978: p. 287). To prove the second part of the result, we observe that H_0 is clearly minimum-phase and asymptotically stable while

$$H(s) = [I + kG(s)]^{-1}G(s)(CB)^{-1},$$

where $G(s) = C(sI - A)^{-1}B$ is the transfer-function matrix of system (1.1). Noting that

$$H_0(s) + H_0^{\mathrm{T}}(\bar{s}) = H_0^{\mathrm{T}}(\bar{s})[H_0^{-1}(s) + (H_0^{\mathrm{T}}(\bar{s}))^{-1}]H_0(s),$$

and setting $\tilde{G} = G(CB)^{-1}$, it is sufficient to show that

$$R(s) := H_0^{-1}(s) + (H_0^{\mathrm{T}}(\bar{s}))^{-1} \equiv kN + \tilde{G}^{-1}(s - \lambda_0) + \tilde{G}^{-1}(\bar{s} - \lambda_0)$$

is nonnegative definite for $s \in \mathbb{C}^+$. Condition (3.4) guarantees the nonsingularity of CB in the above.

Next, write

$$\tilde{G}^{-1}(s - \lambda_0) = (s - \lambda_0)I + H_1(s - \lambda_0),$$

where the transfer-function matrix $H_1(\bullet)$ has poles equal to the zeros of (1.1) and is proper. $H_1(s - \lambda_0)$ is hence uniformly bounded on $\mathbb{C}^+$, with

$$T(s) := H_1(s - \lambda_0) + H_1^{\mathrm{T}}(\bar{s} - \lambda_0) \geqslant -\gamma I \quad (s \in \mathbb{C}^+)$$

for some $\gamma \in \mathbb{R}$. Setting $s = \sigma + \mathrm{i}w$, with $\sigma > 0$, yields, after a little manipulation and denoting the smallest eigenvalue of N by λ_1,

$$R(s) = kN + 2(\sigma - \lambda_0)I + T(s) \geqslant [k\lambda_1 - (2\lambda_0 + \lambda)]I \geqslant 0$$

for all $k \geqslant k^* := (2\lambda_0 + \gamma)/\lambda_1$, which proves the result. □

Using the above, we now state the first main theorem of the paper.

THEOREM 1 *Let the linear system* (1.1) *be of class* $\Sigma(0)$ *with CB satisfying the positivity condition* (3.4), *and denote the largest eigenvalue of N by* $\mathrm{r}(N)$. *If $t \mapsto k(t)$ is a monotonically nondecreasing piecewise continuous function satisfying the divergence conditions* (3.1), *then there exist real, strictly positive numbers* λ,

a_1, a_2 (independent of $k(t)$ and initial condition $\boldsymbol{x}_0$) and a real, strictly positive number M such that, for any system (2.3) *in the class* $C(g_0, h_1, h_2, h_3)$, *where*

$$2h_2 < \mathrm{r}(N), \qquad 1 \geq a_1 g_0 + a_2 h_1, \tag{3.5}$$

the solution $\boldsymbol{x}(t)$ from any initial condition $\boldsymbol{x}_0$ satisfies

$$\|\boldsymbol{x}(t)\| \leq M\mathrm{e}^{-\lambda t}\, \|\boldsymbol{x}_0\| \quad (t \geq 0).$$

Proof. If (1.1) has class $\Sigma(0)$, it is of class $\Sigma(\lambda_0)$ for some $\lambda_0 > 0$. Using the notation of the previous lemmas, choose k^* (Lemma 3) such that $S(A - kBC, B, C)$ has all poles and zeros in the region $\mathbb{C}^{-\lambda_0}$ for all $k \geq k^*$. The matrix H has minimal realization $S\big(A - kBC,\ B(CB)^{-1},\ C\big)$ and H_0 can be supposed to be positive-real for all $k \geq k^*$. Lemma 2 now implies that there exist positive definite symmetric matrices P and Q such that

$$(A - k^*BC)^{\mathrm{T}}P + P(A - k^*BC) = -Q, \qquad PB(CB)^{-1} = C^{\mathrm{T}}. \tag{3.6}$$

Consider the system (2.3) subjected to the control law (1.2), where $k(t)$ is a monotonically increasing piecewise continuous function satisfying (3.1). Clearly there exists $t^* \geq 0$ such that $k(t) \geq k^*$ for all $t \geq t^*$. Writing (2.3) in the form

$$\dot{\boldsymbol{x}} = (A - k^*BC)\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}, t) + B[(k^* - k)\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{x}, -k\boldsymbol{y}, \boldsymbol{y}, t) - \boldsymbol{f}(\boldsymbol{y}, t)]$$

and defining the 'Liapunov' function

$$v_P(\boldsymbol{x}) = \mathrm{tr}\,(\boldsymbol{x}^{\mathrm{T}}P\boldsymbol{x})$$

yields, after a little manipulation,

$$\dot{v}_P(\boldsymbol{x}) = -\mathrm{tr}\,(\boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{x}) + 2\,\mathrm{tr}\,(\boldsymbol{x}^{\mathrm{T}}P\boldsymbol{g}) + 2\,\mathrm{tr}\,\big(\boldsymbol{x}^{\mathrm{T}}PB[(k^* - k)\boldsymbol{y} + \boldsymbol{h} - \boldsymbol{f}]\big) \tag{3.7}$$

(here, and in the sequel, we selectively suppress the arguments of $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$, and k). From (3.7), using (3.6), (2.4), (1.2), and (3.4), and the fact that $v_P(\bullet)$, $v_Q^{\frac{1}{2}}(\bullet)$, $\|\bullet\|$, and $\|\bullet\|_2$ are topologically equivalent norms in $\mathbb{R}^{n\times l}$, we have for $t \geq t^*$:

$$\begin{aligned}
\dot{v}_P(\boldsymbol{x}) &\leq -\mu v_P(\boldsymbol{x}) + 2\,\mathrm{tr}\,(\boldsymbol{x}^{\mathrm{T}}P\boldsymbol{g}) + 2\,\mathrm{tr}\,(\boldsymbol{y}^{\mathrm{T}}CB[(k^* - k)\boldsymbol{y} + \boldsymbol{h} - \boldsymbol{f}]) \\
&\leq -\mu v_P(\boldsymbol{x}) + g_0 m_1 v_P(\boldsymbol{x}) + 2(k^* - k)\,\mathrm{tr}\,(\boldsymbol{y}^{\mathrm{T}}CB\boldsymbol{y}) + 2m_2h_1v_P(\boldsymbol{x}) \\
&\qquad + 2h_2k\,\|\boldsymbol{y}\|_2^2 + 2h_3\,\|\boldsymbol{y}\|_2^2 \\
&\leq -(\mu - g_0m_1 - 2m_2h_1)v_P(\boldsymbol{x}) + 2\,\|\boldsymbol{y}\|_2^2\,[\tfrac{1}{2}\mathrm{r}(N)(k^* - k) + h_2k + h_3],
\end{aligned}$$

for some $\mu > 0$ and positive constants m_1 and m_2. It follows that

$$\dot{v}_P(\boldsymbol{x}) \leq -\lambda v_P(x) \quad (t \geq t^{**}),$$

where $t^{**} \geq t^*$ is a solution of the equation

$$\tfrac{1}{2}\mathrm{r}(N)[k^* - k(t^{**})] + h_2k(t^{**}) + h_3 = 0,$$

provided that $g_0M_1 + 2M_2h_1 < \mu - \lambda$. Choosing $0 < \lambda < \mu$ provides a natural definition of a_1 and a_2 and demonstrates that $x(\bullet)$ is asymptotically exponentially bounded and hence uniformly exponentially bounded as required. This completes the proof of the theorem. □

The important point of the result is that, subject to the specified structural constraints and the specified tolerance to time-varying nonlinearities, monotonic *divergence* of $k(t)$ $(t \to \infty)$ leads to exponential *convergence* of the state trajectory. The explanation for this fact is that sufficiently high gains are achieved to take the system into a configuration where a stabilizing positive-real structural property holds. This interpretation alone makes the above proof more satisfying than those of Byrnes & Willems (1984), Mårtensson (1986), and Ilchmann, Owens, & Prätzel-Wolters (1986). Note that the positive-real approach, as compared with the previous norm-based methods, also permits the inclusion of unbounded nonlinearities (see equations (2.4b,c)). The controllability and observability requirement on $S(A, B, C)$ can be relaxed to stabilizability and detectability, but this is excluded for brevity.

To complete this section, we consider the constraint (3.4) on the validity of Theorem 1 and its apparent limitations on the applicability. To demonstrate that this is not a real problem, we will take the view that the theorem is really about the exponential stability of the linear system (1.1) and its tolerance to nonlinear effects. For the linear system (1.1) in the presence of the control (1.2), the closed-loop dynamics can be represented by

$$\dot{\boldsymbol{x}}(t) = [A - k(t)BC]\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0.$$

Noting that $BC = (BT^{-1})\ (TC)$ for any $m \times m$ nonsingular real matrix T, it is easily seen that the stability analysis is unchanged if $\boldsymbol{v} = T\boldsymbol{u}$ is regarded as controlling $\boldsymbol{w} = T\boldsymbol{y}$ via the feedback law $\boldsymbol{v} = -k\boldsymbol{w}$. This change of input/output variables is equivalent to the map $CB \mapsto TCBT^{-1}$. This is important, because we then have the following simple lemma.

LEMMA 4 *If CB has spectrum $\sigma(CB) \subset \mathbb{C}^+$, then there exists a nonsingular real transformation T such that, with $D := TCBT^{-1}$, we have $D + D^{\mathrm{T}} > 0$.*

Proof. In the case when CB has real eigenvalues $\lambda_1, \ldots, \lambda_m$, it is real-similar to the matrix $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_m) + E$, where the elements of E can be made arbitrarily small. The natural transformation to use is the Jordan canonical-form transformation with upper off-diagonal elements all set equal to ε, where ε is small. The condition $\sigma(CB) \subset \mathbb{C}^+$ ensures that $\lambda_i > 0$ $(i = 1, \ldots, m)$, and the result is trivially verified. In the case of complex eigenvalues, the analysis is straightforward but lengthy. The details are hence omitted for brevity. □

The implications of the lemma for the linear case are that, by using the transformation $(\boldsymbol{u}, \boldsymbol{y}) \mapsto (\boldsymbol{v}, \boldsymbol{w})$ defined earlier, with T obtained from Lemma 4, the proof of Theorem 1 carries through with no change (except for the deletion of the nonlinearities) to provide the following known result.

THEOREM 2 *Let the linear system* (1.1) *be of class $\Sigma(0)$. Then there exist constants $M > 0$ and $\lambda > 0$ such that, if $t \mapsto k(t)$ is any monotonically nondecreasing piecewise continuous function satisfying the divergence condition* (3.1), *then the resultant state trajectory $\boldsymbol{x}(t)$ $(t \geq 0)$, from any initial condition $\boldsymbol{x}_0$, satisfies*

$$\|\boldsymbol{x}(t)\| \leq M\mathrm{e}^{-\lambda t}\, \|\boldsymbol{x}_0\| \quad (t \geq 0).$$

Given this result, the methods of proof of Theorem 1 can be used to demonstrate the tolerance of the result to nonlinear time-dependent perturbations. The class of perturbations is again of the form of (2.4), but expressed in terms of new data and coordinates defined by the map

$$(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y}, CB) \mapsto (\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{w}, D) = (T\boldsymbol{u}, \boldsymbol{x}, T\boldsymbol{y}, TCBT^{-1}).$$

The details are omitted for brevity.

4. A general class of adaptive controllers

In this section, we follow the development of Ilchmann *et al.* (1986) to define a wide class of adaptation mechanisms for nonlinear systems of the form described in Section 2. For simplicity of presentation, we concentrate on the use of Theorem 1, but it is a simple problem to extend the ideas to include the results of Theorem 2 and its following remarks.

Let $S(A, B, C) \in \Sigma(0)$ and consider the class $\mathrm{C}(g_0, h_1, h_2, h_3)$ of systems satisfying the conditions of Section 2 together with conditions (3.4)–(3.5). To construct a wide class of time-varying gain adaptation rules, consider the following linear spaces:

$$\mathrm{C}^{q\times l}[0\,,\infty) := \{z(\bullet) : [0\,,\infty) \to \mathbb{R}^{q\times l} : z(\bullet) \text{ is continuous on } \mathbb{R}^+\},$$
$$\mathrm{E}^{q\times l} := \{z(\bullet) \in \mathrm{C}^{q\times l}[0\,,\infty) : \|z(t)\| < M\mathrm{e}^{-\alpha t} \;\; (t>0) \;\; \text{for some } \alpha, M > 0\}.$$

The required performance of the adaptive scheme is taken to be specified by the requirement $z(\bullet) \in P^{q\times l}$, where $P^{q\times l}$ is a linear space with the property that $\mathrm{E}^{q\times l} \subset P^{q\times l} \subset \mathrm{C}^{q\times l}[0\,,\infty)$. This choice is open to the designer. The set $\hat{\mathrm{K}}(P^{q\times l})$ of adaptive gains $\psi : z(\bullet) \to k(\bullet)$ is defined to be the set of causal maps $\psi : \mathrm{C}^{q\times l}[0\,,\infty) \to \mathrm{C}[0\,,\infty)$ that satisfy the conditions:

$$\text{(a)} \;\; \psi(P^{q\times l}) \subset \mathrm{L}_\infty[0\,,\infty), \qquad \text{(b)} \;\; \psi(z) \in \mathrm{L}_\infty[0\,,\infty) \;\Rightarrow\; z \in P^{q\times l}, \tag{4.1}$$

and (c) $\psi(z)(t)$ is piecewise continuous and monotonically nondecreasing for all $t \geqslant 0$ and every $z \in \mathrm{C}^{q\times l}[0\,,\infty)$. Examples of such maps include the maps ψ defined by

$$\dot{k}(t) = \Phi(\|z(t)\|), \qquad k(0) = k_0;$$

here, $\Phi : \mathbb{R} \to \mathbb{R}$ is any map satisfying $\delta q(\lambda) \geqslant \Phi(\lambda) \geqslant q(\lambda)$ $(\lambda \geqslant 0)$, for some $\delta \geqslant 1$ and some nontrivial polynomial $q(\lambda) = \sum_{i=1}^{\infty} q_i \lambda^i$, with $q_i \geqslant 0$ $(i \geqslant 1)$ and with $q_i > 0$ only for those indices in a finite nonempty set Q. In such a situation, $\psi \in \hat{\mathrm{K}}\,(P^{q\times l})$, where

$$P^{q\times l} = \bigcap_{i\in Q} \mathrm{L}_i^{q\times l}[0\,,\infty).$$

The adaptation (1.3) is hence just one element of the class $\hat{\mathrm{K}}(\mathrm{L}_2^{m\times 1}[0\,,\infty))$. A slightly more general set of maps is

$$\dot{\eta}(t) = \Phi(\|z(t)\|), \qquad \eta(0) = \eta_0, \qquad k(t) = \eta(t) + \varepsilon \max_{0\leqslant t'\leqslant t} \|z(t')\|\,,$$

with $\varepsilon > 0$; then $\psi \in \hat{\mathrm{K}}\ (P^{q\times l})$ with

$$P^{q\times l} = \mathrm{L}^{q\times l}[0\,,\infty) \cap \bigcap_{i\in Q} \mathrm{L}_i^{q\times l}[0\,,\infty).$$

It is clear from the above that it is possible to construct an infinite number of gain adaptation rules. The following result generalizes that of Ilchmann *et al.* (1986), and shows that every one is capable of stabilization in a well-defined sense.

THEOREM 3 *Let the linear system* (1.1) *be of class* $\Sigma(0)$ *with* CB *satisfying* (3.4) *and measurements* z *given by* (2.4). *Let* $S(A, B, C, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{f})$ *be any system of the form of* (2.3)–(2.4) *in the class* $\mathrm{C}(g_0, h_1, h_2, h_3)$, *where* g_0, h_1, h_2, h_3 *satisfy the conditions* (3.5) *of Theorem* 1. *Then, if* ψ *is any element of* $\hat{\mathrm{K}}\ (P^{q\times l})$, *the control* (1.2) *with gain adaptation* $k = \psi(z)$ *generates a closed-loop system whose response from any initial condition* $\boldsymbol{x}_0$ *has the property* $z(\bullet) \in P^{q\times l}$ *and whose gain* $k(t)$ *satisfies the convergence requirement* (1.4).

Remark (1) The result demonstrates, by the arbitrariness of $\psi \in \hat{\mathrm{K}}\ (P^{q\times l})$, that there is great freedom in choice of adaptation mechanism once the 'target space' $P^{q\times l}$ of required measurement responses is specified. (2) A similar result can be based on Theorem 2 but is omitted for brevity.

Proof. If $\lim_{t\to\infty} k(t) = k_\infty < \infty$, then $\psi(z) \in \mathrm{L}_\infty\ [0\,,\infty)$ and hence, using (4.1b), $z \in P^{q\times l}$ as required. The only other possibility is that $k(t) \to \infty\ (t \to \infty)$; then Theorem 1 states that $\boldsymbol{x}(\bullet) \in \mathrm{E}^{q\times l}$ and hence $z(\bullet) \in \mathrm{E}^{q\times l} \subset P^{q\times l}$. But then $\psi(z) \in \mathrm{L}_\infty[0\,,\infty)$ (by (4.1a)) so that k_∞ exists and is finite. This contradiction completes the proof of the result. □

To complete this section, we note that, if stabilization of *all* systems in $\Sigma(0)$ is required, the above result represents a general design strategy that fits nicely into the requirement that $z(\bullet)$ lies in the intersection of a finite family of L_p spaces. If specific *exponential decay* of $z(t)$ is required, it does not appear to be possible to use this result. If, however, we concentrate on stabilization of a specific system, the following result illustrates the possibility of using adaptive mechanisms that guarantee *exponential* stability. Again, the theorem is based on Theorem 2 for the nonlinear case, assuming condition (3.4) on CB. A similar result for the linear case and arbitrary CB satisfying $\sigma(CB) \subset \mathbb{C}^+$ is easily derived.

THEOREM 4 *Under the conditions of Theorem* 3, *there exists* $\varepsilon^* > 0$ *such that the control* (1.2) *with gain adaptation*

$$k(t) = k_0 + \max_{0\leqslant t'\leqslant t} \mathrm{e}^{\varepsilon t'}\, \|z(t')\|, \tag{4.2}$$

where $0 < \varepsilon < \varepsilon^*$, *generates a closed-loop system whose response from any initial condition* $\boldsymbol{x}_0$ *has the property* $\|z(t)\| \leqslant M(\varepsilon, \boldsymbol{x}_0)\mathrm{e}^{-\varepsilon t}$ $(t \geqslant 0)$, *for some* $M(\varepsilon, \boldsymbol{x}_0) \geqslant 0$, *and the gain* $k(t)$ *satisfies the convergence requirement* (1.4).

Proof. Note that $k(\bullet)$ defined by (4.2) is piecewise continuous and monotonically nondecreasing. Using the notation of Theorem 1, set $\varepsilon^* = \lambda > 0$ and put

$0<\varepsilon<\varepsilon^*$. If k_∞ is finite, then

$$\|z(t)\|<\mathrm{e}^{-\alpha t}(k_\infty-k_0)\quad (t\geq 0),$$

and the result is trivial. The only other possibility is that $k_\infty=\infty$; then Theorem 1 indicates that

$$k(t)\leq k_0+MF_0\max_{0\leq t'\leq t}\mathrm{e}^{(\varepsilon-\lambda)t'}\leq k_0+MF_0\quad (t\geq 0),$$

whence k_∞ is finite (by monotonicity). This contradiction completes the proof of the result. □

The result shows the existence of adaptive mechanisms that produce exponentially bounded responses. Note, however, that we require some knowledge of ε^*, and hence more knowledge of the system's dynamics, to guarantee success.

5. Adaptive high-gain stabilization with gain switching

An underlying requirement of the previous sections is that condition (2.1) holds, which represents the assumption that some knowledge of the 'sign' of the high-frequency gain is available to the designer. If this knowledge is not available, then the control problem becomes more complex. This problem has been examined by Byrnes & Willems (1984) and others in the case of $n=l=1$ with (2.1) relaxed to the assumption that it is only known that $CB\neq 0$. In such a situation, they have shown that the feedback law

$$u(t)=-N_0\big(k(t)\big)k(t)y(t), \tag{5.1}$$

with gain adaptation described by

$$\dot{k}(t)=y^2(t),\qquad k(0)=k_0, \tag{5.2}$$

is capable of producing L_2 response characteristics from the closed-loop system. The important new ingredient in this control strategy is the *switching function* $N_0(\bullet)$, dependent upon the current gain $k(t)$ which ensures stability if it is a (so-called) *Nussbaum gain* e.g. a scalar map $N_0:\mathbb{R}^+\to\mathbb{R}$ satisfying the conditions

$$\limsup_{k\to\infty} k^{-1}\int_0^k N_0(\mu)\mu\,\mathrm{d}\mu=\infty,\qquad \liminf_{k\to\infty} k^{-1}\int_0^k N_0(\mu)\mu\,\mathrm{d}\mu=-\infty.$$

It is the purpose of this section to consider the application of the positive-real analysis as a mechanism for:

(i) extending the results of Willems & Byrnes (1984) to include the time-varying nonlinear elements described in Section 2;

(ii) enabling the switching strategy to depend upon current and past data records of gain and output behaviour (a strategy that, intuitively, should be capable of improving the adaptive mechanism);

(iii) making possible the use of more-general gain adaptations than that described by (5.2).

Throughout the section, we will retain the nomenclature and notation of the previous sections, but now with $m = l = 1$ and assumption (2.1) replaced by the assumption that $CB \neq 0$. The control law will be of a slightly more general form than (5.1), namely,

$$u(t) = -\mathrm{N}(s(t))k(t)y(t), \tag{5.3}$$

where $k(t)$ represents the gain adaptation, $\mathrm{N} : \mathbb{R}^+ \to \mathbb{R}$ is a continuous map with the property that, for some $a \geq 0$,

$$\limsup_{\sigma\to\infty} \frac{1}{\sigma - a}\int_a^\sigma \mathrm{N}(\mu)\,\mathrm{d}\mu = \infty, \qquad \liminf_{\sigma\to\infty} \frac{1}{\sigma - a}\int_a^\sigma \mathrm{N}(\mu)\,\mathrm{d}\mu = -\infty, \tag{5.4a,b}$$

and $s(t)$ is a signal defined in the following lemma. It is easily seen that, if conditions (5.4) are satisfied for some $a \geq 0$, they are satisfied for all $a \geq 0$.

LEMMA 5 *Let the linear system* (1.1) *be of class* $\Sigma(0)$. *Let* $k(t)$ *be any strictly positive, monotonically nondecreasing, piecewise continuous function on* $\mathbb{R}^+$, *and let* $u(\bullet)$ *be generated from the feedback mechanism* (5.3) *with* $s(t)$ *generated from the differential equation*

$$\dot{s}(t) = k(t)y^2(t), \qquad s(0) = s_0, \tag{5.5}$$

with $s_0 \geq 0$. *Then there exist real, strictly positive numbers* a_1 *and* a_2 *(independent of* $k(t)$, $\boldsymbol{x}_0$, *and* s_0*) such that any system* (2.3) *in the class* $\mathrm{C}(g_0, h_1, h_2, h_3)$, *with*

$$h_2 = 0, \qquad 1 \geq a_1 g_0 + a_2 h_1, \qquad h_3 = 0, \tag{5.6}$$

has a closed-loop response $\boldsymbol{x}(\bullet) \in \mathrm{L}_p^n[0, \infty)$ $(2 \leq p \leq \infty)$ *with*

$$\lim_{t\to\infty} s(t) = s_\infty < \infty. \tag{5.7}$$

Proof. If $\boldsymbol{x}_0 = \mathbf{0}$, the proof is trivial, since $\boldsymbol{x}(t) \equiv \mathbf{0}$ $(t \geq 0)$. Suppose therefore that $\boldsymbol{x}_0 \neq \mathbf{0}$ and hence, by observability, $y(t)$ has only isolated zeros and so the map $t \mapsto s(t)$ is strictly monotonically increasing and has a strictly monotonically increasing, continuous inverse. Let $\delta = \mathrm{sgn}\,(CB) = CB/|CB|$ and choose $\lambda_0 > 0$ so that (1.1) is in $\Sigma(\lambda_0)$. The conditions of Lemma 3 now hold true for $S(A, \delta B, C)$, so we can deduce the existence of $k^* \geq 0$ such that $S(A - k^*B\delta C, \delta B, C)$ is asymptotically stable for all $k \geq k^*$ with H_0 positive-real for $k \geq k^*$. Lemma 2 then guarantees the existence of symmetric positive definite matrices P and Q such that

$$(A - k^*B\,\delta C)^{\mathrm{T}}P + P(A - k^*B\,\delta C) = -Q, \qquad PB(CB)^{-1} = C^{\mathrm{T}}.$$

Using the Liapunov function $v(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}P\boldsymbol{x}$ and applying the method of the proof of Theorem 1 yields, for suitable choice of a_1 and a_2,

$$\dot{v}(\boldsymbol{x}) \leq -\lambda v(\boldsymbol{x}) + y^2CB[\delta k^* - \mathrm{N}(s)k] \quad (t \geq 0),$$

where $\lambda > 0$. Integration of this inequality over the interval $[0, t]$ yields, after a little manipulation,

$$\begin{aligned}0 &\leq v(\boldsymbol{x}(t)) + \lambda \int_0^t v(\boldsymbol{x}(t'))\,\mathrm{d}t' \\ &\leq v(\boldsymbol{x}_0) + CB\Big(\delta k^* \int_0^t y^2(t')\,\mathrm{d}t - \int_0^t N(s(t'))k(t')y^2(t')\,\mathrm{d}t'\Big) \\ &\leq v(\boldsymbol{x}_0) + CB\Big(\frac{\delta k^*}{k(0)} \int_0^t k(t')y^2(t')\,\mathrm{d}t' - \int_0^t N(s(t'))k(t')y^2(t')\,\mathrm{d}t\Big).\end{aligned} \tag{5.8}$$

Writing

$$\int_0^t k(t')y^2(t')\,\mathrm{d}t' = s(t) - s_0$$

and using the properties of Riemann–Stieltjes integrals, we have

$$\int_0^t N(s(t'))k(t')y^2(t')\,\mathrm{d}t' = \int_0^t N(s(t'))\dot{s}(t')\,\mathrm{d}t' = \int_0^t N(s(t'))\,\mathrm{d}s(t') = \int_{s_0}^{s(t)} N(\mu)\,\mathrm{d}\mu;$$

then equation (5.8) becomes

$$\begin{aligned}0 &\leq v(\boldsymbol{x}(t)) + \lambda \int_0^t v(\boldsymbol{x}(t'))\,\mathrm{d}t' \\ &\leq v(\boldsymbol{x}_0) + CB[s(t) - s_0]\Big(\frac{\delta k^*}{k(0)} - \frac{1}{s(t)-s_0}\int_{s_0}^{s(t)} N(\mu)\,\mathrm{d}\mu\Big).\end{aligned}$$

Application of properties (5.4) of $N(\bullet)$ then, in a similar manner to Willems & Byrnes (1984), proves that (5.7) holds true. It follows that there exists $M > 0$ such that for all $t \geq 0$

$$v(x(t)) + \lambda \int_0^t v(x(t'))\,\mathrm{d}t' \leq M.$$

Hence, since $v^{\frac{1}{2}}(\boldsymbol{x})$ is a norm on $\mathbb{R}^n$, we obtain

$$\boldsymbol{x}(\bullet) \in \mathrm{L}^n_\infty[0,\infty) \cap \mathrm{L}^n_2[0,\infty) \subset \mathrm{L}^n_p[0,\infty) \quad (2 \leq p \leq \infty).$$

This completes the proof of the lemma. □

The above lemma should be regarded as a generalization of that of Byrnes & Willems, which is just the special case obtained by setting

$$k(t) \equiv 1 \quad (t \geq 0), \qquad \boldsymbol{g} = \boldsymbol{h} = \boldsymbol{f} = \boldsymbol{0}, \qquad N(\lambda) = N_0(\lambda)\lambda,$$

with N_0 a Nussbaum gain, and noting that $y(\bullet) \in \mathrm{L}_2[0,\infty)$. The main contributions of the lemma therefore are:

(i) the inclusion of a well defined class of nonlinearities, hence supporting the notion that the switching policy is robust with respect to plant dynamics;

(ii) the proof that $\boldsymbol{x}(\bullet) \in \mathrm{L}^n_p[0,\infty)$ $(2 \leq p \leq \infty)$;

(iii) the demonstration that the switching mechanism $N(\bullet)$ can depend upon past and current data records on output y and gain k, convergence of the switching mechanism being independent of the gain adaptation mechanism for generating $k(t)$.

The results of Lemma 5 do not require the boundedness of $k(t)$, nor do they appear to imply the convergence of $N(s(t))k(t)$ as $t\to\infty$. The natural practical requirement is hence to ensure that condition (1.4) holds. Lemma 5 provides an infinite number of possibilities in this direction. For example, the feedback strategy makes it natural to regard $k(\bullet)$ as the image of a causal map ψ operating on data records for $s(\bullet)$, $y(\bullet)$, $z(\bullet)$. In this circumstance, the set $\hat{K}_s$ of adaptive gains

$$\psi : (s(\bullet), y(\bullet), z(\bullet)) \mapsto k(\bullet)$$

can be defined to be the set of causal maps

$\{\psi : X \to L_\infty[0, \infty)$: X is a finite or infinite intersection of product spaces of the form $L_\infty[0,\infty) \times L_{r_1}[0,\infty) \times L^q_{r_2}[0,\infty)$, with $2 \leqslant r_i \leqslant \infty$ ($i = 1, 2$), and ψ is such that $\psi(s, y, z)(t)$ is strictly positive, piecewise continuous, and monotonically nondecreasing on $[0,\infty)\}$.

In these circumstances, we obtain the following result.

THEOREM 5 *Let the linear system* (1.1) *be of class* $\Sigma(0)$. *Choose* $\psi \in \hat{K}_s$ *and consider the application of the feedback mechanism* (5.3) *with* $s(t)$ *generated by* (5.5) *and* $k(t) = \psi(s, y, z)(t)$. *Then, there exist strictly positive real numbers* a_1 *and* a_2 (*independent of* ψ, $\boldsymbol{x}_0$, *and* s_0) *such that any system* (2.3) *in the class* $C(g_0, h_1, h_2, h_3)$, *with* (5.6) *satisfied, has a closed-loop response*

$$\boldsymbol{x}(\bullet) \in L^n_p[0,\infty) \quad (2 \leqslant p \leqslant \infty),$$

with

$$\lim_{t\to\infty} s(t) = s_\infty < \infty, \qquad \lim_{t\to\infty} k(t) = k_\infty < \infty. \tag{5.9a,b}$$

Proof. The properties of $x(\bullet)$ and $s(\bullet)$ follow from Lemma 5, and it is consequently clear that $s(\bullet) \in L_\infty[0,\infty)$, $y(\bullet) \in L_p[0,\infty)$ $(2 \leqslant p \leqslant \infty)$, and $z(\bullet) \in L^q_p[0,\infty)$ $(2 \leqslant p \leqslant \infty)$. Hence $\psi(s, y, z) \in L_\infty(0,\infty)$, and (5.9b) follows from the monotonicity assumption. □

The theorem contains all the necessary ingredients of convergence and stability. That is, stability of the state response and convergence of the switching and gain adaptation mechanisms. The important point to note is that this result is true for a very large class of control laws and hence represents a major extension of previous work.

To conclude this section, the wide scope of gain adaptation mechanisms is illustrated by the following example of $\psi \in \hat{K}_s$:

$$\dot{\eta}(t) = y^2(t)\phi_1(s(t), y(t)) + \|z(t)\|^2\, \phi_2(s(t), \|z(t)\|), \qquad k(t) = k_0 + \eta(t),$$

with $\eta(0) + k_0 > 0$, and ϕ_i $(i = 1, 2)$ finite polynomials with positive coefficients.

The precise use of such a large number of possibilities will be the subject of future studies.

6. The case of $CB > 0$ revisited

The analysis of Section 5 indicates that the theory of high-gain stabilization of Sections 3 and 4 can be further extended. The basic ideas of this extension are outlined in this section, but the degrees of freedom made available and their possible use in design are left for further study and will be reported in a future paper. One of the underlying assumptions in the switching-free case is that the gain is monotonically nondecreasing. To indicate that this requirement can be relaxed, it is only necessary to point out that, in the case of $n = l = 1$, the proof of Lemma 5 requires that $N(\bullet)$ must satisfy *both* conditions (5.4), since the sign of CB is assumed not to be known. If, however, it is known that $CB > 0$, then condition (5.4b) is not required for the results of Lemma 5, and hence Theorem 5, to be valid. Since as $N(\bullet)$ is not necessarily *monotonic,* the control law

$$u(t) = -N\big(s(t)\big)k(t)y(t)$$

need not have a monotonic gain $N(s)k$ variation. The potential practical benefits of this result are that the possibility of *gain reduction* could reduce the value of the converged stabilizing gain, and hence offset the tendency of high-gain adaptation mechanisms to use higher gains than is absolutely necessary.

7. Conclusions

Recent results on high-gain adaptive stabilization fit naturally into the system-theoretic structures of positive-realness. The consequences of positive-real structure for the existence of solutions to augmented Liapunov equations make possible (a) the inclusion of nonlinear time-varying perturbations (hence demonstrating the robustness of the theory), (b) the construction of a wide class of adaptation mechanisms, and (c) the extension of high-gain switching algorithms to separate the switching mechanism from the gain adaptation.

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